LECTURE 4.

- Equivalence relations
- Equivalence classes
- Partial orders

Definition.

Suppose R is an equivalence relation on X. For every element $a \in X$ the *equivalence class* of *a* is the set $[a]_R = \{x \in \mathrm{X} | aRx\}$.

Examples.

1. $[2]_{\equiv_5} = \{5k + 2 | k \in \mathbb{Z}\}$ – the set of all integers that yield 2 as the remainder of division by 5. Notice that $[2]_{\equiv_5} = [7]_{\equiv_5} = [-3]_{\equiv_5}$ etc.
2. The equivalence class of {1} of the equipotency relation on $2^{\mathbb{R}}$ is the set of all single-element subsets of ℝ. In this case every equivalence class is a set of sets. Writing [5] makes no sense in this context. 5 is not an element of $2^{\mathbb{R}}$.
3. Equivalence classes of ∥ are called *directions*. Every line parallel to a given line *l* shares its direction with *l*.

When it is clear from the context what relation we have in mind we drop the label of the relation in $[a]_R$ and we write simply [a].

Definition.

A family of subsets $\{A_i\}_{i \in I}$ of X is called a *partition of X* iff

- $(\forall i, j \in I)(i \neq j \Rightarrow A_i \cap A_j = \emptyset)$ (the sets are *pairwise disjoint*)
- $\bigcup_{i \in I} A_i = X$ (the family covers X)

For example, the set of all lines passing through the origin is NOT a partition of the plane because they have a common point (0,0). Since (0,0) is the only common point we can construct another family of sets: the set of all lines passing through the origin, each without the point (0,0), and the single element set {(0,0)}.

The set of all lines parallel to the horizontal axis OX is a partition of the plane.

Lemma.

If R is an equivalence relation on X, then

for every $a, b \in X, [a] = [b]$ iff aRb.

Proof.(⇒)
Since $b \in [b]$ and $[a] = [b]$, $b \in [a]$ which means aRb.

(⇐) Since aRb, $b \in [a]$. For every $x \in [b], bRx$. From transitivity of R we obtain that aRx $i.e.,$ $x \in [a]$. This means $[b] \subseteq [a]$. In the same way we can show that $[a] \subseteq [b]$.

Theorem.

For every equivalence relation R on X the set of all equivalence classes of R is a partition of X.

Proof.

For every $a \in X, a \in [a]_R$ hence, X is covered by equivalence classes of R.

Suppose that equivalence classes are not pairwise disjoint. Then, there exist a and b in X such that $[a] \neq [b]$ and there exists $p, p \in [a] \cap [b]$. This means aRp and bRp. Since R is symmetric and transitive, we obtain aRb so, by lemma, $[a] = [b]$.
A contradiction. QED

The last theorem can be, in a sense, reversed:

Theorem.
For every partition P = $\{A_i\}_{i \in I}$ of *X* there exists an equivalence relation *R* on *X* such that P is the set of equivalence classes of *R*.

Proof.
Let us define a relation *R* as follows:
xRy iff $(\exists i \in I)(x \in A_i \wedge y \in A_i)$

We must prove that:

(a) *R* an equivalence relation

(b) $\{[x]_R \colon x \in X\}$ = P.

(a) *R* is obviously reflexive (because P covers *X*) and symmetric. Suppose xRy and yRz. Then $x, y \in A_i$ and $y, z \in A_j$ for some $i, j \in I$. Since the sets from P are pairwise disjoint, $y \in A_j$ and $y \in A_i$ we get *i*=*j* which means *xRz* *i.e.*, *R* is transitive.

(b) For every x, $(\exists i \in I)\ x \in A_i$. Clearly, $[x] = A_i$. Hence, $\{[x]_R : x \in X\} \subseteq$ P. On the other hand, given any $i \in I$, and any $x \in A_i$, A_i is the equivalence class for x, which means P $\subseteq$ $\{[x]_R : x \in X\}$. QED

The remainder of this slide and the next two is just some propaganda. You can read it or ignore it.

FAQ 1. *What is so exciting about equivalence relations*?

They organize our thinking about the (mathematical) universe. For example we can define a relation between sets: two sets A and B are related iff there is a bijection (1-1 and 'onto' function mapping A onto B). Then we develop the concept of a natural number saying that a "number" is the common property of all sets in one equivalence class of this relation, or that a natural number IS an equivalence class of this relation.

FAQ 2. *Is it only me or is the last proof completely stupid? You define a relation with pre-defined equivalence classes (sets forming the partition) by saying that two elements are related whenever they belong to the same set and then you proudly announce that related elements belong to the same set. W.T.H. does it tell us about the nature of the connection between related elements? WTH does it tell us about Universe? It is just a formal trick!*

This is one of many counterexamples to the *excluded middle* logical principle because the answer is both YES and NO:

'**NO**' because we do not classify proofs into silly-looking and sophisticated but into correct and faulty. From where I stand the proof is correct and therefore cannot be shrugged off as 'stupid'.

‘**YES**’ because it IS only a formal trick and it doesn’t tell us a thing about the Universe. Rather, it tells us something about *the way we look* at the Universe. In the actual life, given a partition of a set we would like to discover a ‘proper reason’ for an element to belong to a particular class. For example you can divide people into those who do not contract COVID-19, those who do but survive and those who do and die. Of course it is useless to say ‘the common property of humans who belong to one of these sets is that they do belong to the same set’, even though it does the trick, it defines an equivalence relation. What we are after is a more analytical or cause-oriented definition of the relation, like two people are in the same class because they share a specific sequence in their DNA.

PARTIAL ORDERS

Definition 1. A *partial order* on a set X is any relation R on X that is reflexive, transitive and antisymmetric. The pair (X,R) is called a *poset* (as in *p*artially *o*rdered *set*).

The concept of a partial order is modelled on the ≤ relation (LEQ) for numbers or, better but less popular among the general public, "$\subseteq$", the inclusion relation on 2^X for some set X. "$\subseteq$" is better because, one way or the other, ≤ happens between every two numbers (a≤b or b≤a) and there is nothing in the definition of a partial order to suggest that this should be the case. The inclusion on the other hand does not have this property, one can easily find sets which are “incomparable”, like $\{1,2\}$ and $\{1,3\}$, so this order is more ‘partial’ (‘partial’ as opposed to ‘total’).

People tend to denote partial orders by symbols whose graphical form implies something ‘directional’, i.e. $\leftarrow$ is better than $\approx$, or $\neg$ is better than $\perp$. Or the curly $\preccurlyeq$ symbol or just plain $<$ symbol. It must be clear from the context that you mean a partial order on a set not the usual ‘less than’ relation.

Examples.

1. EQ, the equality relation on any set X is obviously reflexive, transitive and antisymmetric (symmetric as well). This is a very egalitarian example, no element of X is 'greater than', or 'older than' another. In other words no two different elements are comparable.
2. On the other extreme we have the 'complete' order, $R=X\times X$, where everybody is 'above' everybody else. But, do we? Is it antisymmetric? NO, this is not a partial order.
3. $(\mathbb{N}, |)$
4. $(2^X, \preccurlyeq)$ with $A \preccurlyeq B$ meaning that $|A| \leq |B|$? NO, this relation is not antisymmetric if $|X| \geq 2$.

We often illustrate posets using Hasse diagrams where elements of X are represented by dots or small circles, the higher the dot is in your picture, the higher the corresponding element of X is in $(X, \preccurlyeq)$. Comparable elements are supposed to be joined by lines; but self-loops and lines whose existence can be deduced from transitivity are omitted to keep the picture readable.

The Hasse diagram should not be confused with the *"graph of a relation*". In the graph we include **all** lines connecting related elements.

Example.

The Hasse diagram for $(2^X, \subseteq)$ where X = {x,y,z}. Here, ovals representing subsets are labelled with the actual subsets and instead of lines we use arrows pointing from "smaller" to "larger" elements.

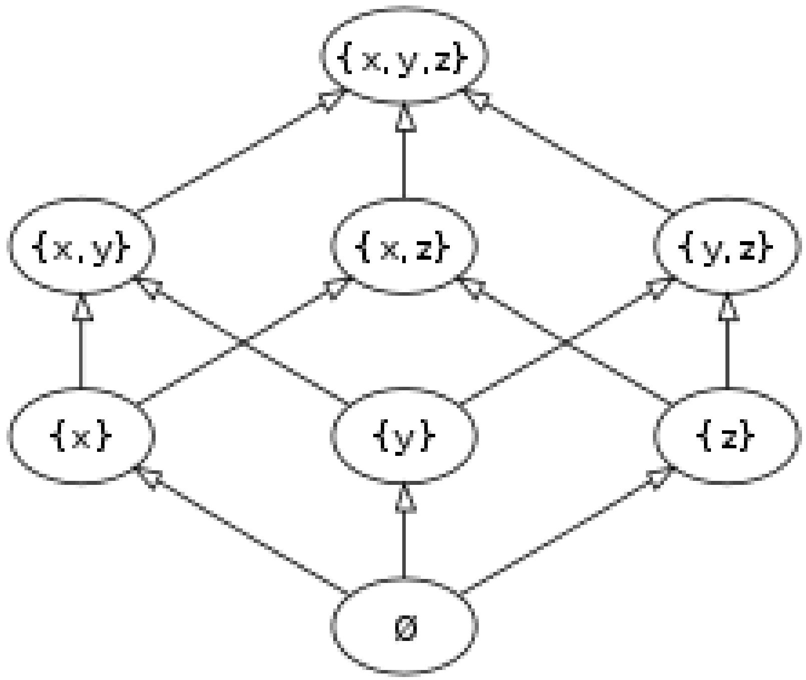

Illustration from Wikipedia

Comprehension.

1. Sketch the Hasse diagram for $(\{1,2,3, \ldots , 12\}, \mid)$, where $a|b$ means "*a* divides *b*".
2. Let X be a set. EQ on X turned out to be both an equivalence relation on X and a partial order on X. Can you find other relations with this property?

Definition2. Let $(X, \preccurlyeq)$ be a poset, $p \in X$.

(a) p is a *largest* element of $(X, \preccurlyeq)$ iff $(\forall x \in X)\ x \preccurlyeq$ p

(b) p is a *smallest* element of $(X, \preccurlyeq)$ iff $(\forall x \in X)\ p \preccurlyeq x$

(c) p is a *maximal* element of $(X, \preccurlyeq)$ iff

$$(\forall x \in X)(p \preccurlyeq x => p = x)$$

(d) p is a *minimal* element of (X, $\preccurlyeq$) iff

$$(\forall x \in X)(x \preccurlyeq p => p = x)$$

You should notice that *largest* means “everybody else is below *p*” while *maximal* means “there is nobody above *p*”. There is a difference if the order is *partial*. Similar remark applies to *smallest* and *minimal*.

Notice that in (X, EQ) every element is both minimal and maximal and there is no smallest and no largest element – unless X is a one-element set, in which case the only element of X is minimal, maximal, smallest and largest at the same time.

Comprehension.

1. Prove that the largest element in a poset (if there is one) is the only maximal element.
2. Is it true that, if a poset has exactly one maximal element *m* then *m* is the largest element as well?